The Generic Multiple-Precision Floating-Point Addition With Correct Rounding

(as in the MPFR Library)

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Introduction

MPFR: Arbitrary-precision floating-point system in base 2.

Considered here: the addition of numbers having the **same sign**.

- The addition of floating-point numbers: a “simple” operation, easy to understand? But **many different cases** for the generic addition (with arbitrary precisions).
- In MPFR, the addition had been buggy for a long time (missing particular cases…), despite several patches.
  $\rightarrow$ I completely rewrote the addition function (October 2001).
- How about the complexity? Seems obvious, but…

The MPFR Floating-Point Addition

Note: The negative case is obtained from the positive case.

Input:

- **Positive numbers x and y** of resp. precisions $m \geq 2$ and $n \geq 2$.
- **Target precision** $p \geq 2$.
- **Rounding mode $\diamond$** (to $-\infty$, to $+\infty$, to 0, or to the nearest).

Output:

- $\diamond_p(x+y)$, i.e. correctly-rounded result.
- **Sign** of $\diamond_p(x+y) - (x+y)$, called *ternary value*.

The Floating-Point Representation

- All the values considered here are positive real numbers.
- Floating-point representation in precision p:

$$0.b_1b_2b_3\ldots b_p \times 2^e$$

  where the b_i's are binary digits (0 or 1) forming the *mantissa* and e is the *exponent* (a bounded integer).
- The representation is *normalized*: $b_1 \neq 0$, i.e. $b_1 = 1$.
- We do not consider *subnormals* here (MPFR does not support them).

Computation Steps

The addition (without considering optimization) consists in:

1. ordering x and y so that $e_x \geq e_y$,
2. computing the exponent difference $d = e_x - e_y$,
3. shifting the mantissa of y by d positions to the right,
4. initializing the exponent e of the result to e_x (temporary value),
5. adding the mantissa of x and the shifted mantissa of y (shifting the result by 1 position to the right and incrementing e if there is a carry),
6. rounding the result (setting the mantissa to 0.1 and incrementing e if a carry is generated due to an upward rounding).

Exponent Considerations

- Assume $e_x \geq e_y$.
- Addition of the aligned mantissas with rounding, with 1 or 2 possible carries (due to rounding and arbitrary precision, e.g. $0.111 + 0.111$ gives 0.10×2^2 for $p = 2$, rounding upwards).
- Exponent $e_{x+y} = e_x +$ carries.

Underflow: impossible.

Possible overflow, but no practical or theoretical difficulties.
$\rightarrow$ Will not be considered here (i.e. assume unbounded exponents).

$\rightarrow$ We now concentrate on the addition of the mantissas.

Rounding an Exact Real Value

Canonical infinite mantissa of the exact result: $0.1b_2b_3b_4b_5\ldots$

The rounding can be expressed as a function of the rounding mode, the **rounding bit** $r = b_{p+1}$ and the **sticky bit** $s = b_{p+2} \vee b_{p+3} \vee \ldots$

r / s	downwards	upwards	to the nearest
0 / 0	exact	exact	exact
0 / 1	$-$	$+$	$-$
1 / 0	$-$	$+$	$-$ / $+$
1 / 1	$-$	$+$	$+$

"$-$" means: exact mantissa truncated to precision p.

"$+$" means: add 2^{-p} to the truncated mantissa ($\rightarrow$ possible carry).

Finding an Efficient Algorithm

Trailing bits of x and/or y may have no influence on the result.
For instance:

$$0.1010100000010010001 + 0.10001 \times 2^{-9}$$

rounded to 4 bits.

Only the first 6 bits 101010 of x (and none for y) are necessary to deduce the result and the ternary value.

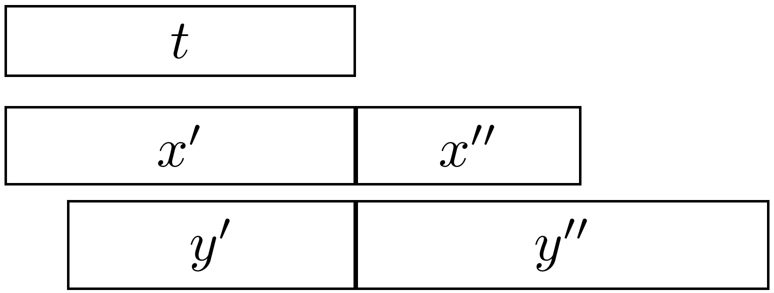
The goal: **take into account as few input bits as possible**.

Note: bits are grouped into words in memory. To simplify, we give here a bit-based description of the algorithm.

The addition can be written $x + y = t + \varepsilon$, where

- t (*main term*) is computed with the first $p + 2$ bits of x and the corresponding $\max(p + 2 - d, 0)$ bits of y,
- ε (*error term*) satisfies $0 \leq \varepsilon < 2^{e_x - p - 1} \leq (1/2)\,\mathrm{ulp}(x + y)$, with equality if there are no carries.

Graphically:



where x'' may be empty and either y' or y'' may be empty (and x'' may end after y'', and if y' is empty, y'' may start after x'' ends).

Computing the Main Term

The *main term* t is computed and written in **time $\Theta(p)$**:

- an $\Omega(p)$ time is necessary to fill the $p+2$ bits;
- a linear time is obviously sufficient.

Note: different ways to compute the main term, due to different overlappings and trailing zeros (see the paper for the details concerning the MPFR implementation).

Possible carry detection (to avoid a separate shift) by looking at the most significant bits of x and y first (not implemented in MPFR).

Special bits: $\begin{cases} \text{Bit } p+1\text{: temporary } \textit{rounding bit } r_t. \\ \text{Bit } p+2\text{: } \textit{following bit } f. \end{cases}$

If a Carry Was Generated...

Then $p + 3$ bits of the result have really been computed (instead of $p + 2$).

$\rightarrow$ In the implementation, consider that the bit $p + 3$ comes from the first iteration of the processing described in a few slides and must be taken into account accordingly.

$\rightarrow$ In the following tables, we may assume that $p + 2$ bits of the result have been computed and the bit $p + 3$ is part of the error term.

Following Bit and Error → Rounding and Sticky Bits

Let u denote the *weight* $2^{-(p+2)}$ of the bit $p+2$ (*following bit*).
So, $0 \leq \varepsilon < 2u$.

f	ε	r	s	example
0	$\varepsilon = 0$	$=$	0	$1000_r 0_f + 0.0000$
0	$\varepsilon > 0$	$=$	1	$1000_r 0_f + 1.1101$
1	$\varepsilon < u$	$=$	1	$1000_r 1_f + 0.1101$
1	$\varepsilon = u$	$+$	0	$1111_r 1_f + 1.0000$
1	$\varepsilon > u$	$+$	1	$1000_r 1_f + 1.0001$

"$=$" means: the rounding bit is the temporary rounding bit $p+1$.
"$+$" means: 1 must be added to the temporary rounding bit $p+1$.

r_t	f	ε	r	s	downwards	upwards	to the nearest
0	0	$\varepsilon = 0$	0	0	exact	exact	exact
0	0	$\varepsilon > 0$	0	1	$-$	$+$	$-$
0	1	$\varepsilon < u$	0	1	$-$	$+$	$-$
0	1	$\varepsilon = u$	1	0	$-$	$+$	$-$ / $+$
0	1	$\varepsilon > u$	1	1	$-$	$+$	$+$
1	0	$\varepsilon = 0$	1	0	$-$	$+$	$-$ / $+$
1	0	$\varepsilon > 0$	1	1	$-$	$+$	$+$
1	1	$\varepsilon < u$	1	1	$-$	$+$	$+$
1	1	$\varepsilon = u$	0	0	exact	exact	exact
1	1	$\varepsilon > u$	0	1	$-$	$+$	$-$

Iteration Over the Remaining Bits

Assume one iterates over bits $p + 3$, $p + 4$, $p + 5 \ldots$ (best solution?).

At each iteration, the mantissa of the temporary result has the form: $0.1z_2z_3 \ldots z_prfff \ldots fff$ with an error in the interval $[0, 2)$ ulp, and one iterates as long as the bits after the (temporary) rounding bit are identical.

- $f = 0$: while $x_i = y_{i-d} = 0$.
- $f = 1$: while $x_i + y_{i-d} = 1$. If $x_i = y_{i-d} = 1$, then point $f = 0$.

Particular case: y hasn't been read yet, i.e. $d \geq p + 2$.
If $f = 0$, take into account the fact that $y_1 = 1$: $s = 1$.

The Complexity

We assume that:

- the mantissa bits are 0 and 1 with equal probabilities,
- x and y are independent numbers.

Time complexity in $\Omega(p)$ and in $O(m+n+p)$.
Worst case in $\Theta(m+n+p)$. Average case in $\Theta(p)$.

In some cases: many possible orders to test the trailing bits.

Note: As the natural distribution of the real numbers is logarithmic, in a *very theoretical* point of view, it is better to start with the least significant bits for the 0 equality test (i.e. when $f=0$).

The MPFR Implementation

- Bits grouped into *limbs* (32-bit or 64-bit unsigned integer).
- Bit-based algorithm → limb-based algorithm (not difficult, but more cases to deal with!).
- Bits $p+1$ and $p+2$ in variables `rb` and `fb`, determined on the fly, as soon as they are known (again, many cases...).
- In addition to the p bits of the target, more bits may be taken into account for the main term (to fill the least significant limb).

Various cases in the main term computation; in particular: whether d is a multiple of the limb size. Very dependent on the GMP functions.

Various cases for the error term:

- x'' has not entirely been read and y'' has not been read yet.
- x'' and y'' overlap.
- x'' has not entirely been read and y'' has entirely been read.
- x'' has entirely been read and y'' has not been read yet.
- x'' has entirely been read and y'' has not entirely been read.
- x'' and y'' have entirely been read.

In the overlapping case: two limbs are added. The loop ends as soon as the result is different from 0 for $f = 0$ or the maximum limb value `MP_LIMB_T_MAX` for $f = 1$.

Conclusion

- Not so simple, after all...
- The (bit-based) theoretical analysis could help to improve the current MPFR implementation.
- The theoretical analysis could also be useful to provide a full mechanically-checked proof.
- Future work: deal with the subtraction, but more difficult (e.g. possible cancellation, when subtracting very close numbers).